Modified Thin Shock Layer Method for Supersonic Flow Past an Oscillating Cone

W.H. Hui* and G.E. Mahood†
University of Waterloo, Waterloo, Ont., Canada

The problem of supersonic flow past a circular cone oscillating about its vertex is considered. The amplitude and the frequency parameter of the oscillation are assumed to be small, and a perturbation solution in the amplitude and frequency is sought. Furthermore, thin shock layer expansion is used to derive the flowfield solution in the form of a series. The first three terms in the series are obtained, showing that the series solution tends to converge when the shock layer is very thin and otherwise it tends to diverge. The technique of parameter straining then is applied which greatly improves the accuracy and extends the range of applicability of the thin shock layer solution. In particular, simple explicit formulas for the stability derivatives are valid for moderate as well as high freestream Mach numbers and for thick as well as slender cones. Variations of the stability derivatives with the freestream Mach number, specific heat ratio, and the cone semiangle are investigated and comparisons with existing theories are included. The relation of limiting gasdynamic theory with unsteady Newtonian flow theory also is discussed.

Nomenclature

j	$=1+(1/\lambda M_\infty^2 \sin^2\tau)$
k	$=\omega\ell/U_\infty$
ℓ	= cone length, taken as unity
M_∞	= freestream Mach number
m	$=\tan\tau$
$\hat{n}$	= unit vector normal to cone surface
$\bar{p}$	= pressure, normalized by $\rho_\infty U_\infty^2$
r	= radial coordinate, normalized by ℓ
$\bar{t}$	= time
t	$=U_\infty \bar{t}/\ell$
$\bar{u},\bar{v},\bar{w}$	= velocity component in r, ϑ, ϕ direction, respectively, normalized by U_∞
U_∞	= freestream velocity
$\bar{\rho}$	= density, normalized by ρ_∞
ρ_∞	= freestream density
τ	= semivertex angle of cone
γ	= ratio of specific heats
ϵ	$=(\gamma-1)/(\gamma+1)+(1/M_\infty^2\sin^2\tau)$
λ	$=(\gamma-1)/(\gamma+1)$
$\bar{\theta}$	= amplitude of oscillation
$\theta(t)$	$=\bar{\theta}e^{ikt}$
ω	= circular frequency of oscillation

I. Introduction

THE problem of steady supersonic flow with attached shock wave past a circular cone at zero incidence is described by the well-known Taylor-Macoll equations and has been solved numerically by several authors.[1-3] Some approximate analytic solutions also have been found such as those obtained for slender cones at low supersonic speeds,[4] and those valid in the hypersonic range using the constant density assumption.[5] Recently, the thin shock layer method, together with parameter straining, has been used to derive an improved formula for the surface pressure that remains accurate across the whole Mach number range for which the shock remains attached to the cone vertex.[6]

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*Associate Professor, Department of Applied Mathematics.

†Graduate Student, Department of Applied Mathematics.

The flow about a circular cone that has been placed at a small angle of attack to the freestream was studied by Stone,[7] and numerical solutions obtained by Kopal[8] and Sims[9] for a limited number of combinations of the freestream Mach number M_∞, the cone semiangle τ, and the specific heat ratio γ. Although Ferri[10] pointed out that Stone's straight forward perturbation is not uniformly valid because of a thin layer of intense vorticity near the cone surface, Munson[11] later showed that such an expansion still gives uniformly valid results to the first order for the circumferenctial velocity and to any order (in angle of attack) for the pressure. Doty and Rasmussen[12] derived an approximate analytic solution to the perturbation equations under constant density and hypersonic small-disturbance assumptions. Also, for low supersonic speeds, Van Dyke[13] has developed a "second-order" potential theory, in which the exact tangency condition and isentropic pressure relation are satisfied. Relatively few studies have been done on unsteady flows past circular cones and analytical results particularly have been limited. Tobak and Wehrend[14] have extended Van Dyke's potential theory for the inclined cone to the problems of pitching and plunging cones at low supersonic speeds. Brong[15] solved numerically the first-order perturbation equations for the pitching and plunging cones and presented some results for a few combinations of τ and γ. Since the oscillating motion of a cone about its vertex can be regarded as a combination of pitching and plunging motions, the results of the above papers also can be adapted to the case of oscillating cones. McIntosh[16] has perturbed the hypersonic small-disturbance equations to obtain a numerical solution to the oscillating cone problem, valid for slender cones at large values of M_∞. Orlik-Ruckemann[17] has used a curvefitting analysis to present an accurate formula for McIntosh's results.

The present paper examines the supersonic flow with attached shock wave past a circular cone performing small amplitude slow oscillations about its vertex. The mean position of the cone is at zero angle of attack. By perturbing about the amplitude and frequency parameter, the first-order in-phase and out-of-phase equations are derived which then are solved by a further expansion of the dependent and independent variables as power series in ϵ. The solution is carried out to the third term in ϵ, and the static and dynamic stability derivatives $C_{N\theta}$ and $C_{N\dot{\theta}}$ are calculated. Then these are compared to previous numerical and analytical results and, in both cases, the stability derivatives are shown to converge rapidly for $\epsilon << 1$, but to diverge for large values of ϵ. However, the ac-

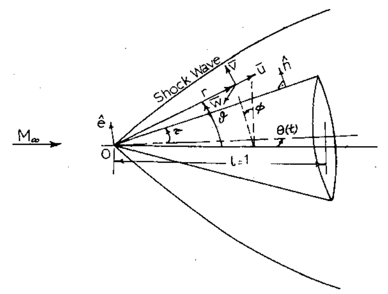

Fig. 1 Configuration showing notation.

curacy and the range of validity of both formulas for $C_{N\theta}$ and $C_{N\dot\theta}$ are improved greatly and extended by suitably applying Pritulo's parameter straining technique.[18]

The analytic solution presented in this paper possesses an important advantage over the previous analysis in that it is accurate over a wide range of the parameters γ, M_∞, and τ. Computer solution are limited necessarily to only a few values of these parameters, and previous analytical work has been restricted to potential flow, constant density, or hypersonic small-disturbance assumptions. In contrast to the hypersonic small-disturbance results, which are valid only for small τ, the accuracy of these formulas actually improves for large values of τ. The usefulness of the strained parameter technique also is demonstrated, as the accuracy of the strained formulas is extended to values of M_∞ much closer to one and to values of τ much smaller than could be expected from the original thin shock layer expansion.

II. Perturbation Equations and Solutions

Perturbation Equations

To study the problem of supersonic flow past an oscillating circular cone, the wind-fixed spherical coordinates (r,ϑ,ϕ) are used (Fig. 1) in which $r=o$ is the cone vertex, $\vartheta=o$ coincides with the freestream direction, and the plane $\phi=o$ is normal to the axis of oscillation. For stability analysis, both the amplitude of oscillation $\bar\theta$ and the frequency parameter k are assumed small. The equation of the cone surface is then

$$\vartheta=\tau+\bar\theta e^{ikt}\cos\phi+O(\bar\theta^2) \tag{1}$$

A first-order perturbation solution in $\bar\theta$ and k is sought in the form[7,15]

$$\bar u=u(\vartheta)+\bar\theta e^{ikt}[U(\vartheta)+ikr\tilde U(\vartheta)+O((ik)^2)]\cos\phi+O(\bar\theta^2) \tag{2a}$$

$$\bar v=v(\vartheta)+\bar\theta e^{ikt}[V(\vartheta)+ikr\tilde V(\vartheta)+O((ik)^2]\cos\phi+O(\bar\theta^2) \tag{2b}$$

$$\bar w=\bar\theta e^{ikt}[W(\vartheta)+ikr\tilde W(\vartheta)+O((ik)^2)]\sin\phi+O(\bar\theta^2) \tag{2c}$$

$$\bar p=p(\vartheta)+\bar\theta e^{ikt}[P(\vartheta)+ikr\tilde P(\vartheta)+O((ik)^2)]\cos\phi+O(\bar\theta^2) \tag{2d}$$

$$\bar\rho=\rho(\vartheta)+\bar\theta e^{ikt}[R(\vartheta)+ikr\tilde R(\vartheta)+O((ik)^2)]\cos\phi+O(\bar\theta^2) \tag{2e}$$

The shock equation is written as

$$\vartheta=\beta+\bar\theta e^{ikt}\{G+ikr\hat G+O[(ik)^2]\}\cos\phi+O(\bar\theta^2) \tag{3}$$

where $U,\tilde U,V,\tilde V$, etc. are real function of ϑ. The constants $\beta,G,\hat G$ must be determined as part of the solution, β being the shock angle of the axisymmetric flow when $\bar\theta=0$. The real part of the expression in braces represents the first-order in-phase flow; the imaginary terms describe the out-of-phase flow. Convergence of the expansion in (ik) of the first-order terms has been shown to be quite rapid,[15,16] and so terms of $O[(ik)^2]$ and higher contribute negligibly to either the in-phase or out-of-phase flow even when k is not necessarily very small.

Substituting (1-3) into the equations of motion for nonviscous nonheat conducting gas[19] and comparing like terms in $\bar\theta$ and k, we obtain a set of equations for the zeroth- and first-order variables. The zeroth-order equations for u,v,w,p,ρ correspond to the Taylor-Macoll equations for the axisymmetric flow past a cone at zero incidence $(\theta(t)\equiv 0)$.[1] The first-order flow quantities for the in-phase flow satisfy the following system of equations:[20]

$$(d/d\vartheta)(\rho V+vR)+(\rho W/\sin\vartheta)+2(uR+\rho U)$$
$$+(\rho V+vR)\cot\vartheta=0 \tag{4a$_i$}$$

$$v\frac{dV}{d\vartheta}+\frac{dv}{d\vartheta}V+uV+vU+\frac{1}{\rho}\frac{dP}{d\vartheta}-\frac{1}{\rho^2}\frac{dp}{d\vartheta}R=0 \tag{4a$_{ii}$}$$

$$v(dW/d\vartheta)+(u+v\cot\vartheta)W-(1/\rho\sin\vartheta)P=0,$$

$$(dU/d\vartheta)-V=0 \tag{4a$_{iii}$}$$

$$(d/d\vartheta)[(P/p)-\gamma R/\rho]=0 \tag{4a$_{iv}$}$$

with the linearized boundary conditions to be satisfied at the mean positions of the body surface and the shock when $\bar\theta=0$

at $$\vartheta=\tau,\quad V=-(dv/d\vartheta) \tag{4b$_i$}$$

at $$\vartheta=\beta,\ \rho V+vR=-[\cos\beta+(d/d\vartheta)(\rho v)]G \tag{4b$_{ii}$}$$

$$P-\sin\beta V=[\sin2\beta-(d/d\vartheta)(p-v\sin\vartheta)]G \tag{4b$_{iii}$}$$

$$U=-[\sin\beta+v]G \tag{4b$_{iv}$}$$

for $$W=(1/\sin\beta)[\sin\beta+v]G \tag{4b$_v$}$$

$$\frac{\gamma}{\gamma-1}P-\frac{\gamma}{\gamma-1}\frac{p}{\rho}R-\sin\beta V=\rho[\sin\beta\cos\beta$$
$$-\frac{d}{d\vartheta}\left(\frac{v^2}{2}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)\Big]G \tag{4b$_{vi}$}$$

The solutions to Eq. (4) correspond to the complete first-order solution in the special case of steady flow past an inclined cone (i.e. when $k=0$).

Finally, the first-order out-of-phase flow quantities which arise because of the unsteady nature of the flow are calculated from the following system of equations:[20]

$$(d/d\vartheta)(\rho\tilde V+v\tilde R)+3(\rho\tilde U+u\tilde R)+(\rho\tilde V+v\tilde R)\cot\vartheta$$
$$+(\rho\tilde W/\sin\vartheta)+R=0 \tag{5a$_i$}$$

$$v(d\tilde U/d\vartheta)-v\tilde V+u\tilde U+(\tilde P/\rho)+U=0 \tag{5a$_{ii}$}$$

$$v\frac{d\tilde V}{d\vartheta}+\left[v\tilde U+\left(u+\frac{dv}{d\vartheta}\right)\tilde V\right]$$
$$+\frac{1}{\rho}\frac{d\tilde P}{d\vartheta}-\frac{1}{\rho^2}\frac{dp}{d\theta}\tilde R+u\tilde V+V=0 \tag{5a$_{iii}$}$$

$$v(d\tilde W/d\varphi)+(u+v\cot\varphi)\tilde W-(\tilde P/\rho\sin\varphi)+u\tilde W=W=0 \tag{5a$_{iv}$}$$

$$v\frac{d}{d\varphi}\left(\frac{\tilde P}{p}-\gamma\frac{\tilde R}{\rho}\right)+u\left(\frac{\tilde P}{p}-\gamma\frac{\tilde R}{\rho}\right)+\frac{P}{p}-\gamma\frac{R}{\rho}=0 \tag{5a$_v$}$$

with the linearized boundary conditions to be satisfied at the mean positions of the body surface and the shock wave when $\bar\theta=0$.

at $$\vartheta=\tau, \quad \tilde{V}=1 \tag{5b$_i$}$$

$$\text{at } \vartheta=\beta, \quad \rho\tilde{V}+v\tilde{R}=[(\rho-2)\cos\beta$$
$$-(d/d\vartheta)(\rho v)]\tilde{G}+(\rho-1)G \tag{5b$_{ii}$}$$

$$\tilde{P}-2\sin\beta\tilde{V}+v^2\tilde{R}=[\sin 2\beta-(d/d\vartheta)(\rho v^2+p)]\tilde{G} \tag{5b$_{iii}$}$$

$$\tilde{U}=-2(\sin\beta+v)\tilde{G}, \tag{5b$_{iv}$}$$

for $$W=(1/\sin\beta)(\sin\beta+\nu)\tilde{G}, \tag{5b$_v$}$$

$$\frac{\gamma}{\gamma-1}\tilde{P}-\frac{\gamma}{\gamma-1}\frac{p}{\rho}\tilde{R}-\sin\beta\tilde{V}$$
$$=\rho\left[\sin\beta\cos\beta-\frac{d}{d\vartheta}\left(\frac{v^2}{2}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)\right]\tilde{G}$$
$$+(\rho-1)\sin\beta\cos\beta\tilde{G}+(\rho-1)\sin\beta G \tag{5b$_{vi}$}$$

Equations (4) and (5) will be solved by the Newtonian expansion based on the assumption that

$$\epsilon<<1 \quad \text{and} \quad j=\text{constant} \tag{6}$$

The quantity ϵ represents the density ratio across the shock which coincides with the cone surface at its mean position. It also characterizes the thickness of the shock layer. As in Ref. 6, the new independent variable μ was introduced by

$$\vartheta=\tau+\epsilon a\mu(1+\epsilon b+\epsilon^2c+\ldots) \tag{7}$$

where the constants a,b,c were determined successively so that the shock angle of the axisymmetric flow corresponded to $\mu=1$, i.e.,

$$\beta=\tau+\epsilon a+\epsilon^2ab+\epsilon^3ac+\ldots \tag{8}$$

In the double limit, $\gamma\to 1$ and $M_\infty\to\infty$ separately (hence $\epsilon\to$ o), the flow is the well-known Newtonian flow. The expansion of the zeroth-order dependent variables, considered as functions of μ and ϵ, thus took the form

$$u=\cos\tau\sum_{i=o}^{\infty}\epsilon^iu_i, \quad v=\sum_{i=1}^{\infty}\epsilon^iv_i$$
$$p=\sin^2\tau\sum_{i=0}^{\infty}\epsilon^ip_i, \quad \rho=(1/\epsilon)\sum_{i=o}^{\infty}\epsilon^i\rho_i \tag{9}$$

The coefficients p_n, etc. were determined successively in Ref. 6. In this paper both the in-phase and out-of-phase first-order equations, (4) and (5), are solved similarly.

Solution of the In-Phase Equations

The first-order in-phase variables, with μ considered the independent variable are expanded in the power series,

$$U=G\sum_{i=0}^{\infty}\epsilon^iU_i, \quad V=G\sum_{i=0}^{\infty}\epsilon^iV_i$$
$$W=G\sum_{i=0}^{\infty}\epsilon^iW_i, \quad P=G\sum_{i=0}^{\infty}\epsilon^iP_i$$
$$R=(G/\epsilon)\sum_{i=0}^{\infty}\epsilon^iR_i, \quad G=\sum_{i=0}^{\infty}\epsilon^iG_i \tag{10}$$

Substituting Eqs. (7-10) into (4) and equating like powers of ϵ, we obtain a sequence of problems of ordinary differential equations for the successive determination of P_n, etc., beginning with $n=0$. For example,[20] for $n=0$,

$$\frac{dV_0}{d\mu}=\frac{dU_0}{d\mu}=0, \quad v_1\frac{dV_0}{d\mu}+\left(\frac{dv_1}{d\mu}+\frac{\sin\tau}{2}\right)V_0+\frac{dP_0}{d\mu}=0$$
$$v_1\frac{dW_0}{d\mu}+\frac{\sin\tau}{2}W_0=0, \quad \frac{d}{d\mu}\left(\frac{P_0}{\sin^2\tau}-R_0\right)=0$$

with the boundary conditions to be satisfied at the mean positions

at $$\mu=0,\ G_0V_0=-(2/m)(dv_1/d\mu) \tag{11a}$$

at $$\mu=1,\ V_0=-(2/m)(dv_1/d\mu),\ U_0=-\sin\tau$$
$$P_0-\sin^2\tau R_0=\sin 2\tau/j$$
$$P_0-\sin\tau V_0=\sin 2\tau\left[1-\frac{dp_1}{d\mu}\right]+2\cos\tau dv_1/d\mu,\ W_0=1 \tag{11b}$$

For $n=1,2$, the equations are similar but more complicated.[20] It is noted that the nth sequence of equations for the determination of V_n, P_n, etc. contains up to the $(n+1)$ th terms in the ϵ expansion of the axisymmetric flow variables. Since the axisymmetric solution was carried out only to $n=3$, the expansion scheme for the first-order in-phase variables cannot go further than $n=2$.

Using the axisymmetric solution,[6] the equations can be integrated,[20] and, especially, we obtain

$$P_0=(\mu+2)\sin\tau\cos\tau, \quad G_0=1$$

$$P_1=\left[\frac{11}{12}\mu^3-\frac{4}{15}(1+m^2)\mu^{5/2}+\left(\frac{m^2}{2}-\frac{7}{4}+\frac{1}{2j}\right)\mu^2\right.$$
$$\left.+\left(\frac{3}{2}-\frac{5m^2}{6}-\frac{1}{j}\right)\mu+\frac{11}{10}-\frac{3}{2j}-\frac{9m^2}{10}\right]\sin\tau\cos\tau$$

$$G_1=(m^2-2)/3+(1/j)$$

$$P_2/\sin\tau\cos\tau=D_0+D_2\mu+D_4\mu^2+D_5\mu^{5/2}$$
$$+D_6\mu^3+D_7\mu^{7/2}+D_8\mu^4+D_9\mu^{9/2}+D_{10}\mu^5$$

$$G_2=\frac{493}{630}-\frac{191}{315}m^2+\frac{4m^4}{9}+\frac{2m^2}{3j}-\frac{7}{6j}+\frac{1}{3j^2}+\frac{m^2}{3j^2} \tag{12}$$

where the constants D_i are given in Ref. 20. In particular,

$$D_0=-\frac{2003}{5040}+\frac{17m^2}{2520}-\frac{9m^4}{20}+\frac{3m^2}{4j}-\frac{17}{24j}+\frac{11}{10j^2}-\frac{9m^2}{10j^2}$$

Solution of the Out-of-Phase Equations

The expansion of the out-of-phase variables in powers of ϵ takes the form

$$\tilde{U}=\sum_{i=0}^{\infty}\epsilon^i\tilde{U}_i, \quad \tilde{V}=\sum_{i=0}^{\infty}\epsilon^i\tilde{V}_i$$
$$\tilde{W}=\sum_{i=0}^{\infty}\epsilon^i\tilde{W}_i, \quad \tilde{P}=\sum_{i=0}^{\infty}\epsilon^i\tilde{P}_i$$
$$\tilde{R}=(1/\epsilon)\sum_{i=0}^{\infty}\epsilon^i\tilde{R}_i, \quad \tilde{G}=\sum_{i=0}^{\infty}\epsilon^i\tilde{G}_i \tag{13}$$

The method used to solve the out-of-phase Eqs. (5) is completely analogous to that used for the in-phase Eqs. (4). The

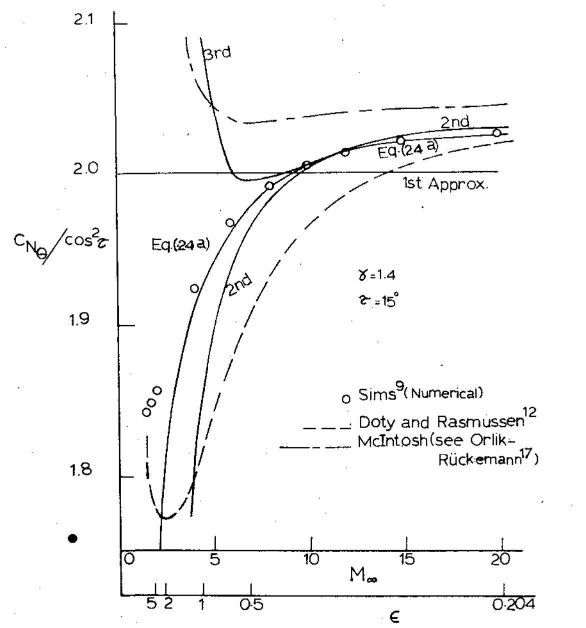

Fig. 2 Variation of $C_{N_\theta}/\cos^2\tau$ with freestream Mach number.

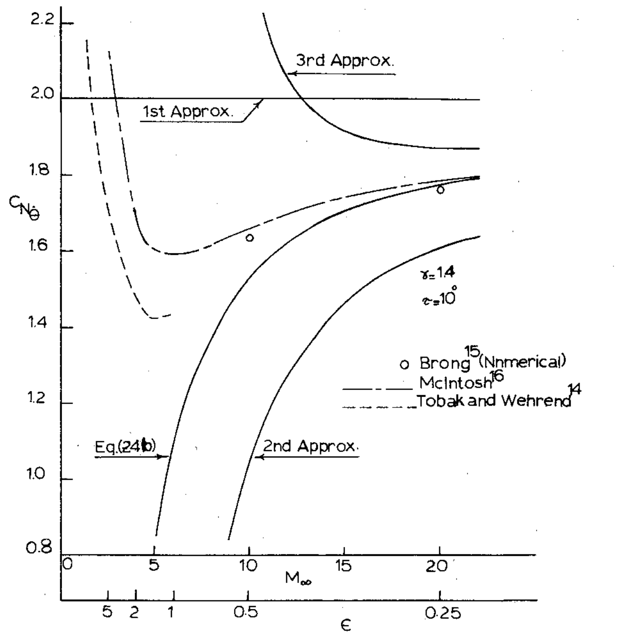

Fig. 3 Variation of $C_{N_{\dot\theta}}$ with freestream Mach number.

results for the out-of-phase pressure is

$$\tilde{P}_0 = (3-\mu)\sin\tau, \qquad \tilde{G}_0 = 0$$

$$\tilde{P}_1 = \sin\tau\Big[-\Big(\frac{1}{12}+\frac{m^2}{6}\Big)\mu^3 + \Big(\frac{2m^2}{3}+\frac{4}{15}+\frac{4}{5j}\Big)\mu^{5/2}$$

$$-\Big(\frac{9}{4}+\frac{3m^2}{4}+\frac{1}{2j}\Big)\mu^2 - \frac{5}{6}\mu - \frac{49}{15} - \frac{m^2}{3} + \frac{41}{30j}\Big]$$

$$\tilde{G}_1 = \frac{1}{\cos\tau}\Big(\frac{1}{3j} - \frac{m^2}{4} - \frac{7}{6}\Big)$$

$$\tilde{P}_2(0) = \Big[\frac{86139}{10080} - \frac{6601}{2520j} - \frac{23}{18j^2} - \frac{757}{1440}m^2$$

$$-\frac{m^2}{3j^2} + \frac{5m^2}{12j} - \frac{731}{720}m^4\Big]\sin\tau$$

$$\tilde{G}_2 = \frac{1}{\cos\tau}\Big[\frac{30647}{10080} - \frac{1367}{2520}m^2 - \frac{1831}{1260j}$$

$$-\frac{7}{18j^2} - \frac{m^2}{9j} - \frac{m^2}{4j^2} - \frac{261}{432}m^4\Big] \tag{14}$$

III. Stability Derivatives and Parameter Straining Technique

Let $\hat{e}$ be a unit vector nomral to the cone axis in the plane $\phi = 0$ (see Fig. 1); the coefficient C_N of the force normal to the cone axis is, according to standard definition,

$$C_N = \frac{\text{Normal force}}{\frac{1}{2}\rho_\infty U_\infty^2 A_b} = \frac{2}{A_b}\iint_{\text{cone surf}} p\hat{n}\cdot\hat{e}\,ds \tag{15}$$

where A_b is the base area of the cone. The static and dynamic stability derivatives C_{N_θ} and $C_{N_{\dot\theta}}$ then are defined, as standard, by

$$C_{N_\theta} = \lim_{\substack{\theta\to 0\\(\dot\theta=0)}} \frac{\partial C_N}{\partial\theta} \qquad C_{N_{\dot\theta}} = \lim_{\substack{\dot\theta\to 0\\(\theta=0)}} \frac{\partial C_N}{\partial\dot\theta} \tag{16}$$

Expanding the surface pressure about $\vartheta = \tau$ [using Eq. (1)], and noting that from the Taylor–Macoll equations $dp/d\vartheta|_{\vartheta=\tau} = 0$, we get the result

$$\tilde{P}_{surf} = \{p + \theta\cos\phi[P + (dp/d\vartheta) + ik\tilde{P} + \ldots] + \ldots\}_{\vartheta=\tau}$$

$$= [p + \theta\cos\phi(P + ik\tilde{P} + \ldots) + \ldots]_{\mu=0} \tag{17}$$

Performing the integration in Eq. (15), ignoring terms of $O(\bar\theta^2)$ and $O(k^2)$, and using (12) and (14), the stability derivatives are found to be given by the series

$$\frac{C_{N_\theta}}{\cos^2\tau} = \frac{P|_{\mu=0}}{\sin\tau\cos\tau} = 2 + \epsilon\Big[\frac{1}{2j} - \frac{7}{30}(1+m^2)\Big] + \epsilon^2$$

$$\Big(\frac{2189}{5040} - \frac{67m^2}{280} + \frac{5m^4}{36} + \frac{41m^2}{60j} - \frac{113}{120j} + \frac{4}{15j^2} - \frac{7m^2}{30j^2}\Big) + \ldots \tag{18a}$$

$$C_{N_{\dot\theta}} = \frac{2}{3}\tilde{P}|_{\mu=0} = 2 + \epsilon\Big(-\frac{98}{45} - \frac{2m^2}{9} + \frac{41}{45j}\Big) + \epsilon^2$$

$$\Big(\frac{86139}{15120} - \frac{6601}{3780j} - \frac{23}{27j^2} - \frac{757m^2}{2160}$$

$$-\frac{2m^2}{9j^2} + \frac{5m^2}{18j} - \frac{731}{1080}m^4\Big) + \ldots \tag{18b}$$

Because of the assumptions used in their derivation, the preceding series are expected to be valid only for $\epsilon << 1$. For larger values of ϵ, the higher order terms in ϵ tend to dominate and both series diverge. These are seen from Figs. 2-4 in comparison with known numerical results. The concept of parameter straining, which is similar to Lighthill's strained coordinates technique, has been developed to extend the range of validity of such series expansions.[18] In order to apply this method, Eqs. (18) are first rewritten as series in λ.

$$C_{N_\theta}/\cos^2\tau = 2 + \lambda g_1(j) + \lambda^2 g_2(j) + \ldots,$$

$$C_{N_{\dot\theta}} = 2 + \lambda\tilde{g}_1(j) + \lambda^2\tilde{g}_2(j) + \ldots \tag{19}$$

where

$$g_1(j) = \frac{1}{2} - \frac{7}{30}(1+m^2)j \tag{20a}$$

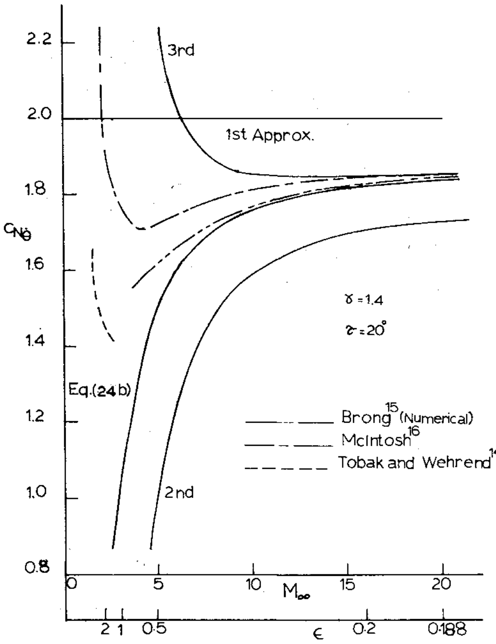

Fig. 4 Variation of $C_{N\dot\theta}$ with freestream Mach number.

$$\tilde{g}_1(j)=\frac{41}{45}-\left(\frac{98}{45}+\frac{2m^2}{9}\right)j \tag{20b}$$

$$g_2(j)=\left(\frac{2189}{5040}-\frac{67}{280}m^2+\frac{5m^4}{36}\right)j^2+\left(\frac{41}{60}m^2-\frac{113}{120}\right)j+\frac{4}{15}-\frac{7}{30}m^2 \tag{20c}$$

$$\tilde{g}_2(j)=\left(\frac{86139}{15120}-\frac{757}{2160}m^2-\frac{731}{10080}m^4\right)j^2+\left(\frac{5m^2}{18}-\frac{6601}{3780}\right)j-\frac{23}{27}-\frac{2}{9}m^2 \tag{20d}$$

The series for $C_{N\theta}$ and $C_{N\dot\theta}$ are divergent for large values of the parameter j (i.e. for small values of $M_\infty \sin\tau$). In each of Eqs. (19), j now is strained slightly by expanding it in a series of the form

$$j=J+\lambda j_1(J)+\lambda^2 j_2(J)+\ldots \tag{21}$$

For example, using Taylor series, the first of Eqs. (19) becomes

$$C_{N_\theta}/\cos^2\tau=2+\lambda G_1(J)+\lambda^2 G_2(J)+\ldots \tag{22}$$

where

$$G_1(J)=g_1(J),\ G_2(J)=g_2(J)+g_1'(J)j_1(J) \tag{23}$$

For $C_{N\theta}$, the straining function $j_1(J)$ is chosen to satisfy the conditions that G_2/G_1 remains bounded as $J\to\infty$ and that $j_1(1)=0$. The latter condition insures that there is no straining for $j=1$ (i.e., $M_\infty=\infty$). The final strained formula for C_{N_θ} is

$$\frac{C_{N_\theta}}{\cos^2\tau}=2+\lambda\left[\frac{1}{2}-\frac{7}{30}(1+m^2)J\right]+\lambda^2\left[-\frac{1213}{5040}+\frac{59}{280}m^2+\frac{5m^4}{36}\right] \tag{24a$_\text{i}$}$$

$$j=J+\lambda\left[\frac{30}{7(1+m^2)}\right]\left[\left(\frac{2189}{5040}-\frac{67}{280}m^2+\frac{5m^4}{36}\right)J^2+\left(\frac{41}{60}m^2-\frac{113}{120}\right)J+\frac{2557}{5040}-\frac{373}{840}m^2-\frac{5m^4}{36}\right] \tag{24a$_\text{ii}$}$$

Similarly, the expression for $C_{N_{\dot\theta}}$ after straining is

$$C_{N_{\dot\theta}}=2+\lambda\left[\frac{41}{45}-\left(\frac{98}{45}+\frac{2m^2}{9}\right)J\right]+\lambda^2\left[\frac{9371}{3024}-\frac{637}{2160}m^2-\frac{731}{10080}m^4\right] \tag{24b$_\text{i}$}$$

$$j=J+\lambda\left[1/\left(\frac{98}{45}+\frac{2m^2}{9}\right)\right]\left[\left(\frac{86139}{15120}-\frac{757}{2160}m^2-\frac{731}{10080}m^4\right)J^2+\left(\frac{5m^2}{18}-\frac{6601}{3780}\right)J-\frac{11947}{3024}+\frac{157}{2160}m^2+\frac{731}{10080}m^4\right] \tag{24b$_\text{ii}$}$$

In Eqs. (24) J can be found in terms of j by solving a quadratic equation, thus giving explicit expressions for $C_{N_{\dot\theta}}$ and $C_{N_{\dot\theta}}$ in terms of the flow parameters (M_∞,γ,τ).

IV. Comparisons with Other Theories and Discussion

Both the strained and the unstrained formulas for the stability derivatives are very easy to use in practice. In Fig. 2 the static stability derivative formulas are compared to Sims' exact numerical results for the sample case $\gamma=1.4$, $\tau=15°$. Results from the hypersonic small-disturbance theories of Doty and Rasmussen[12] and McIntosh[16] also are included. The successive approximate solutions‡ given by the unstrained Eqs. (18a) are seen to converge quite rapidly for $\epsilon<<1$, giving good agreement with the numerical data in the high Mach number regime. However, for larger values of ϵ, the approximations oscillate to diverge as expected. The strained formula considerably extends the accuracy of the thin shock layer method to much smaller values of M_∞. Thus, even for the relatively small cone semiangle of 15°, Eq. (24a) gives significantly better results than does the analytic solution of Doty and Rasmussen[12] based on constant density assumptions. The present formula is also more accurate than the numerical solution of McIntosh.[16,17] (The curve presenting McIntosh's results is actually a plot of $C_{N\theta}/\cos^2\tau$, since terms of $O(\tau^2)$ always are ignored in Ref. 16; otherwise it is further away from Sims' numerical results.) The improvement of the strained formula thus is most pronounced at larger values of τ for which the slender-body assumption of the afore mentioned theories is violated. By contrast, formulas (18a) and (24a) are not restricted to small τ but, in fact, become more accurate as τ increases (and hence ϵ decreases). A result similar to Eq. (18a) for $C_{N\theta}$ up to $O(\lambda)$ also was presented (without analysis, however) by Guiraud [Eq. (10), p. 176)][25] which reads

$$C_{N_\theta}=2\cos^2\tau+\lambda\left(\frac{4}{15}-\frac{1}{2\sin^2\tau}\right)-\frac{7}{30}\,\frac{1}{M_\infty^2\sin^2\tau} \tag{25}$$

It is suspected that there was a typing error in the previous formula and, as a result, the term $-1/(2\sin^2\tau)$ should read $-\tfrac{1}{2}\sin^2\tau$. With this correction, Eq. (25) then would be identical with the first two terms of Eq. (18a) and also would agree with the numerical data presented in Ref. 25.

A more detailed comparison is made in Table 1 of the strained formula for $C_{N\theta}$ with the numerical results of Sims for τ up to 30° and that of Babenko et al.[21] for τ greater than

‡In Figs. 2-4 the nth approximation means the first n terms in the unstrained ϵ series.

Table 1 Initial slope of normal force coefficient C_{N_θ}

M_∞	$\tau=5$	$\tau=10$	$\tau=15$	$\tau=20$	$\tau=30$	$\tau=40$	$\tau=45$
	33.07819	8.45753	3.89872	2.30382	1.16667	ϵ	
2.0	1.09344	1.52984	1.61110	1.58441	1.38390	Eq. (24a)	
	1.91807	1.81834	1.73101	1.63726	1.39887	Numerical	
	14.79401	3.85149	1.82536	1.11651	0.61111	0.43559	
3.0	1.42152	1.69603	1.72425	1.67165	1.44595	1.13373	
	1.89171	1.82030	1.76451	1.68595	1.44971	1.1436	
	5.43251	1.49320	0.76379	0.50861	0.32667	0.26348	0.24667
5.0	1.68289	1.82679	1.81142	1.73690	1.48863	1.16377	0.98866
	1.87157	1.85748	1.81816	1.73907	1.49003	1.1620	0.9893
	1.48313	0.49830	0.31595	0.25215	0.20667	0.19087	0.18667
10.0	1.87587	1.91926	1.86940	1.77735	1.51157	1.17817	1.00026
	1.90759	1.92162	1.87036	1.77815	1.51235		
	0.49578	0.24958	0.20399	0.18804	0.17667	0.17272	0.17167
20.0	1.96663	1.95756	1.89055	1.79063	1.51811	1.18199	1.00328
	1.96902	1.95835	1.89113	1.79109	1.51865		

30°. Data are presented for τ ranging from 5° to 45° and for M_∞ ranging from 2 to 20. The modified thin shock layer method is seen to be accurate to within 2% for values of ϵ as large as 2. This is well outside the range of ϵ for which the original expansion can be applied. Eventually ϵ increases (i.e. either M_∞ or τ decreases) to the point where Eq. (24a) breaks down. Parameter straining thus delays (but does not prevent) the eventual divergence of the stability derivative formulas.

In Figs. 3 and 4 the dynamic stability derivative $C_{N\dot\theta}$ is compared to existing results for $\tau=10°$ and $\tau=20°$, respectively ($\gamma=1.4$). The numerical data are taken from Brong's paper.[15] Orlik-Ruckemann's curve-fitting analysis is used to present further results from McIntosh's numerical solution based on hypersonic small-disturbance theory. The formula for $C_{N\dot\theta}$ given by Orlik-Ruckemann's work[17] is

$$C_{N_{\dot\theta}} = \tfrac{2}{3}(\beta/\tau)^2(-R_2)$$

where R_2 (a function of the hypersonic similarity parameter $K=M_\infty\beta$) is defined in Ref. 17 and β/τ (The ratio of shock angle to cone semiangle at zero angle of attack) can be determined from Sims.[3] Finally, Tobak and Wehrend's results (valid only for M_∞ $\tan\tau<1$) also are given to provide some data for low supersonic speeds. Their stability derivatives for pitching and plunging cones are related to C_{N_θ} and $C_{N_{\dot\theta}}$ by

$$C_{N_\theta} = C_{N_\alpha},\ C_{N_{\dot\theta}} = C_{N_q} + C_{N_{\dot\alpha}} \tag{26}$$

From Figs. 3 and 4 it appears that the successive approximations given by the unstrainded formula (18b) for $C_{N\dot\theta}$ also tend to converge for $\epsilon << 1$, but diverge for larger values of ϵ. The convergence is less rapid than for C_{N_θ}. Again, as seen in Figs. 3 and 4 in comparison with Brong's numerical solution, parameter straining greatly improves and extends the accuracy of the original series. The hypersonic slender-body result works fairly well at large Mach numbers for $\tau=10°$ and even $\tau=20°$, but eventually becomes invalid at larger values of τ. Formulas (18b) and (24b) do not suffer from this handicap.

The thin shock layer solution for $C_{N\theta}$ and $C_{N\dot\theta}$ also can be compared to Newtonian flow theory[5] by taking the limit $\gamma\to 1$, $M_\infty\to\infty$ (i.e. $\epsilon\to 0$). The two approaches are thought to be equivalent in general, but this has been proven only for steady two-dimensional and axisymmetric flows.[22,23] In the unsteady flow case and in the steady asymmetric flow case the situation is more complicated and a note must be made about the meaning of $C_{N\theta}$ and $C_{N\dot\theta}$ as $\epsilon\to 0$. Physically, the problem of finding the stability derivatives of a cone oscillating in a supersonic stream of gas has well-defined and fixed flow parameters (ϵ, M_∞,γ) and, hence, a fixed $\epsilon>0$. The derivatives $C_{N\theta}$ and $C_{N\dot\theta}$ are obtained, as in standard definition like Eq. (16), in the limit

$$\lim_{\bar\theta\to 0} \quad \text{with } \epsilon>0, \text{ fixed} \tag{27a}$$

and in such a limiting process

$$\bar\theta/\epsilon\to 0 \tag{27b}$$

The mathematical solution leading to Eq. (18) or (24) also is consistent§ with this limiting process (27), and C_{N_θ} and $C_{N_{\dot\theta}}$ as given by (24) [or (18)] must be understood as the stability derivatives of an oscillating cone in a given situation for which $\epsilon>$o is kept fixed and $\bar\theta\to$o. Evidently, when ϵ is exactly zero, implying $M_\infty=\infty$, the flow situation cannot be defined physically, nor can $C_{N\theta}$ and $C_{N\dot\theta}$ be calculated by the present method as (27) is not satisfied. On the other hand, for any flow situation such that $\epsilon>0$, no matter how small it may be, the problem of finding the stability derivatives is physically well defined and mathematically solved by (18) [or (24)]. Of course, for smaller $\epsilon>0$, the amplitude of oscillation $\bar\theta$ also must be made much smaller according to (27); but this is compatible with the meaning and definition of the stability derivatives. To summarize: a) $C_{N\theta}$ and $C_{N\dot\theta}$ have no meaning when ϵ is exactly zero, and b) in taking the limit as $\epsilon\to 0$ of $C_{N\theta}$ and $C_{N\dot\theta}$, it also is required that $\bar\theta/\epsilon\to 0$, or more precisely, it means

$$\lim_{\epsilon\to 0}\ \lim_{\bar\theta\to 0} \tag{28}$$

It is in this restricted sense of limiting process that we now compare the limiting values of C_{N_θ} and $C_{N_{\dot\theta}}$ as $\epsilon\to 0$ with Newtonian flow theory. From Eq. (18), as $\epsilon\to 0$

$$C_{N_\theta}/\cos^2\tau\to 2, \quad C_{N_{\dot\theta}}\to 2 \tag{29}$$

Tobak and Wehrend[14] have calculated the stability derivatives using Newtonian impact theory only, ignoring the

§This can be seen as follows. From (1) and (7) at the cone surface $\bar\theta e^{ikt}\cos\phi=\epsilon a\mu+\ldots$. Since the boundary condition (11) on the cone surface is imposed on $\mu=0$ in obtaining (12) and (14), this requires $\bar\theta/\epsilon\to 0$, the same as (27b). Similarly, at the shock surface $\mu=1+0(\bar\theta/\epsilon)$, so imposing the shock boundary conditions (11) at $\mu=1$, as was done in obtaining (12) and (14), again requires (27b).

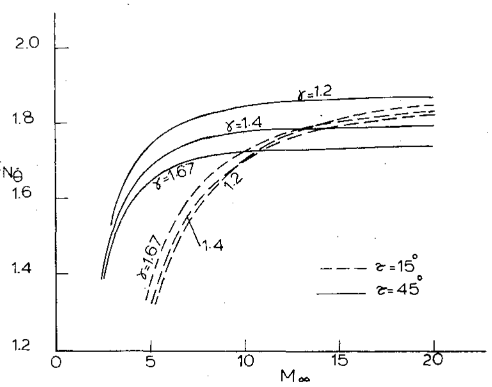

Fig. 5 Effects of λ on $C_{N\dot\theta}$.

centrifugal force contribution. From their results

$$C_{N_\theta}/\cos^2\tau = C_{N_\alpha} - 2,\; C_{N_{\dot\theta}} = C_{N_q} + C_{N_{\dot\alpha}} \to 4/3 + 0 = 4/3 \quad (30)$$

Data similar to Eq. (30) were also presented by Brong[15] for Newtonian flow theory. The conflict of such large difference in the values of $C_{N\dot\theta}$ was resolved in Ref. 24 where it was shown that the centrifugal force terms, which were ignored in Tobak and Wehrend,[14] contribute significantly to the out-of-phase pressure, and that, once they are included, the Newtonian value for $C_{N\dot\theta}$ also is found to be 2, in agreement with the present theory.

$C_{N\dot\theta}$ is plotted in Fig. 5 for two values of τ (15°and 45°) and three values of γ (1.2, 7/5, and 5/3). The necessary data are found easily using the strained formula for $C_{N\dot\theta}$, but would have been difficult (and costly) to obtain by the other numerical methods. The dependence of $C_{N\dot\theta}$ on γ and τ is quite complicated, but one easily observable feature is that the effect of γ on $C_{N\dot\theta}$ is insignificant for slender bodies but becomes much more important for thicker cones at high Mach numbers, in that range $C_{N\dot\theta}$, and hence the dynamic stability of the cone increases with decreasing γ.

V. Conclusions

Using thin shock layer expansions, together with parameter straining, accurate explicit analytical formulas have been obtained for the static and dynamic stability derivatives of a cone oscillating about its vertex. The derived formulas are valid for moderate Mach numbers as well as at hypersonic speeds, and for thick as well as slender cones. By comparison, existing numerical solutions have been tabulated for only a very limited number of combinations of the flow parameters, and previous analytical work has relied on assumptions which lead to a greatly restricted range of applicability. It also is shown that the limiting values of the dynamic stability derivative of oscillating cones as the flow Mach number tends to infinity and the specific heat ratio approaches unity, agree exactly with unsteady Newtonian flow theory when the centrifugal effects are included. Finally, the power of the strained parameter technique clearly is demonstrated as the original series solution, valid only when the shock layer is extremely thin, is extended to moderately thick shock layer cases.

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